Series : SSO/1

Code No. 65/1/1/D

Roll No. | X | X | X | X | X | X | X |

Candidates must write the Code on the title page of the answer-book.

- Please check that this question paper contains 15 printed pages.
- Code number given on the right hand side of the question paper should be written on the title page of the answer-book by the candidate.
- Please check that this question paper contains 26 questions.
- **Please write down the Serial Number of the questions before attempting it.**
- 15 minutes time has been allotted to read this question paper. The question paper will be distributed at 10.15 a.m. From 10.15 a.m. to 10.30 a.m., the student will read the question paper only and will not write any answer on the answer script during this period.

Mathematics

[Time allowed : 3 hours] ***[Maximum marks : 100]***

General Instructions:

(*i*) All questions are compulsory.

(*ii*) The question paper consists of 26 questions.

(*iii*) Marks for each questions are indicated against it.

(*iv*) Questions 1 to 6 in Section-**A** are Very Short Answer Type Questions carrying **one** mark each.

(*v*) Questions 7 to 19 in Section-**B** are Long Answer **I** Type Questions carryying **4** marks each.

(*vi*) Questions 20 to 26 in Section-**C** are Long Answer **II** Type Questions carryying **6** marks each

(*vii*) Please write down the serial number of the Question before attempting it.

1. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

Sol. Projection of $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \dfrac{7\times 2 + 1\times 6 + (-4)\times 3}{\sqrt{2^2 + 6^2 + 3^2}}$

$= \dfrac{14 + 6 - 12}{\sqrt{4 + 36 + 9}}$

$= \dfrac{8}{7}$

2. Find λ, if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

Sol. $\vec{a}, \vec{b}$ and $\vec{c}$ will be co-planar if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ 2 & -1 & -1 \end{vmatrix}$

$= (-3 + 1)i - (-1 - 2)j + (-1 - 6)k$

$= -2\hat{i} + 3\hat{j} - 7\hat{k}$

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a}\times\vec{b}\times\vec{c} = \left(-2\hat{i} + 3\hat{j} - 7\hat{k}\right).(\lambda j + 3k)$

$= 3\lambda - 21$

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$\Rightarrow \quad 3\lambda - 21 = 0$

$\Rightarrow \quad \lambda = 7$

3. If a line makes angles 90°, 60° and θ with *x*, *y* and *z*-axis respectively, where θ is acute, then find θ.

Sol. Since line makes angles 90°, 60° and θ with *x*, *y* and *z*-axis.

$\therefore$ *dr* of the line is cos 90°, cos 60° and cos θ.

$\therefore \quad \cos^2 90° + \cos^2 60° + \cos^2\theta = 1$

$\Rightarrow \quad \dfrac{1}{4} + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = \dfrac{3}{4}$

$\Rightarrow \quad \cos\theta = \dfrac{\sqrt{3}}{2}$ $\qquad \Rightarrow \theta = 30°$

4. Write the element a_{23} of a 3 × 3 matrix A = (a_{ij}) whose elements a_{ij} are given by $a_{ij} = \dfrac{|i - j|}{2}$.

Sol. $a_{ij} = \dfrac{|i - j|}{2}$

$a_{23} = \dfrac{|2 - 3|}{2} = \dfrac{1}{2}$

5. Find the differential equation representing the family of curves $\upsilon = \frac{A}{r} + B$, where A and B are arbitrary constants.

Sol. $\upsilon = \frac{A}{r} + B$

$\upsilon r = A + Br$

Difference both sides w.r.t. r

$$r\frac{d\upsilon}{dr} + \upsilon = B$$

Difference again w.r.t. υ

$$r\frac{d^2\upsilon}{dr^2} + \frac{d\upsilon}{dr} + \frac{d\upsilon}{dr} = 0$$

$$r\frac{d^2\upsilon}{dr^2} + 2\frac{d\upsilon}{dr} = 0$$

6. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dy}{dx} = 1$.

Sol. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ rearranging the given equation

$$\frac{dx}{dy} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is a linear differential equation with P = $\frac{1}{\sqrt{x}}$ and Q = $\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$\therefore$ Integration factor = $e^{\int \frac{dx}{\sqrt{x}}}$

$= e^{\int x^{\frac{-1}{2}} dx}$

$= e^{2\sqrt{x}}$

7. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$.

Sol. $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

$$A^2 = A,\ A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$5A = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$4I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^2 - 5A + 4I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 2 \\ 9 & +2 & 5 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & +5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & +4 & 2 \end{bmatrix}$$

$A^2 - 5A + 4I + X = 0$

$X = -A^2 + 5A - 4I$

$$X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

OR

7. If $A = \begin{bmatrix} 0 & -2 & 3 \\ 0 & -1 & -4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

Sol. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$(A')^{-1} = \frac{adj\ A'}{|A'|}$$

$|A'| = 1(-9) + 0 - 2(-8+3)$

$= -9 + 10 = 1$

$$(adj\ A') = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$(A')^{-1} = \frac{adj\ A'}{|A'|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

8. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants find the value of $f(2x) - f(x)$.

Sol. $f(x) = a\begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

$$= a\begin{vmatrix} 1 & 0 & 0 \\ x & x+a & -1 \\ x^2 & x^2+ax & a \end{vmatrix}$$

$= a[ax + a^2 + x^2 + ax]$

$= a[x^2 + 2ax + a^2]$

$= a(x + a)^2$

$f(2x) = a(2x + a)^2$

$f(2x) - f(x) = a[4x^2 + a^2 + 4ax - x^2 - a^2 - 2ax]$

$= a[3x^2 + 2ax]$

$= ax[3x + 2a]$

9. Find : $\int \frac{dx}{\sin x + \sin 2x}$

Sol. $\int \frac{dx}{\sin x + \sin 2x}$

$$= \int \frac{\sin x dx}{\sin^2 x(1+2\cos x)} = \int \frac{\sin x dx}{(1-\cos^2 x)(1+2\cos x)}$$

Let $\cos x = t$

$-\sin x\, dx = dt$

$$= \quad -\int \frac{dt}{(1+t)(1-t)(1+2t)}$$

Let $\frac{1}{(1+t)(1-t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$1 = A(1 + t)(1 + 2t) + B(1 - t)(1 + 2t) + C(1 + t)(1 - t)$

Solving we get,

$A = \frac{1}{6}$

$B = -\frac{1}{2}$ show it

$C = \frac{4}{3}$

$$\therefore \quad -\int \frac{dt}{(1+t)(1-t)(1+2t)} = -\frac{1}{6}\int \frac{dt}{1-t} - \frac{1}{2}\int \frac{dt}{1+t} - \frac{4}{3}\int \frac{dt}{1+2t}$$

$$= \frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2t| + C$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$

OR

9. Integrate the following w.r.t. x

$$\frac{x^2 - 3x + 1}{\sqrt{1-x^2}}$$

Sol. $\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx$

$$= \int \frac{x^2}{\sqrt{1-x^2}}dx - \int \frac{3x}{\sqrt{1-x^2}}dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \int \frac{x^2-1+1}{\sqrt{1-x^2}}dx + \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{3x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{x^2-1}{\sqrt{1-x^2}}dx + \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{3x}{\sqrt{1-x^2}}dx$$

$$= -\int \sqrt{1-x^2}dx + 2\int \frac{dx}{\sqrt{1-x^2}} - \int \frac{3x}{\sqrt{1-x^2}}dx$$

$$= -\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right] + 2\sin^{-1}x - \int \frac{3x}{\sqrt{1-x^2}}dx$$

$$= -\frac{x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1}x - \int \frac{3x}{\sqrt{1-x^2}}dx$$

$$= -\frac{x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1}x + 3\sqrt{1-x^2} + C$$

10. Evaluate : $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

Sol. $\int_{-\pi}^{\pi} [\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx]\, dx$

$$= \int_{-\pi}^{\pi} \cos^2 ax\, dx + \int_{-\pi}^{\pi} \sin^2 bx\, dx - 2\int_{-\pi}^{\pi} \cos ax . \sin bx\, dx$$

$$= 2\int_{0}^{\pi} \cos^2 ax\, dx + 2\int_{0}^{\pi} \sin^2 bx\, dx - 0$$ [$\int_{-a}^{a} f(x)dx = 0$, , Since $f(x)$ is odd function]

$$= 2\int_{0}^{\pi} \frac{1+\cos 2ax}{2}dx + 2\int_{0}^{\pi} \frac{1-\cos 2bx}{2}dx$$

$$= \left[x + \frac{\sin 2ax}{2a}\right]_0^{\pi} + \left[x - \frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$= 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

11. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.

Sol. A_1 = 1st bag is chosen

A_2 = 2nd bag is chosen

B = One red and one black ball is drawn.

$P(A_1 \cap B) + P(A_2 \cap B)$

$= P(B/A_1)\, P(A_1) + P(B/A_2)\, P(A_2)$

$$= \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} \times \frac{1}{3} + \frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} \times \frac{2}{3}$$

$$= \frac{4\times 6}{45} \times \frac{1}{3} + \frac{7\times 3}{45} \times \frac{2}{3}$$

$$= \frac{8}{45} + \frac{14}{45}$$

$$= \frac{22}{45}$$

OR

11. An unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

Sol. Probability distribution function

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

first find out the probability separately then fill the table

$$E(x) = 0.\frac{1}{16} + 1.\frac{4}{16} + 2.\frac{6}{16} + 3.\frac{4}{16} + 4.\frac{1}{16}$$

$$= \frac{4+12+12+4}{16} = \frac{32}{16} = 2$$

$$E(x^2) = 0 \times \frac{1}{16} + 1^2 \times \frac{4}{16} + 2^2 \times \frac{6}{16} + 3^2 \times \frac{4}{16} + 4^2 \times \frac{1}{16}$$

$$= \frac{0+4+24+36+16}{16} = \frac{80}{10} = 5$$

Mean = E (x) = 2

Variance = E $(x^2) - \{E(x)\}^2 = 5 - 2^2 = 1$

12. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $(\hat{r} \times \hat{i}).(\hat{r} \times \hat{j}) + xy$

Sol. $= (\hat{r} \times \hat{i}).(\hat{r} \times \hat{j}) + xy$

$= (z\hat{j} - y\hat{k}).(x\hat{k} - z\hat{i}) + xy$

$= -xy + xy$

$= 0$

13. Find the distance between the point (–1, –5, –10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

Sol. $x\hat{i} + y\hat{j} + z\hat{k} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$

$x = 3\lambda + 2$; $y = 4\lambda - 1$, $z = 12\lambda + 2$

$3\lambda + 2 - (4\lambda - 1) + 12\lambda + 2 = 5$

$3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 5$

$11\lambda + 5 = 5$

$\lambda = 0$

$x = 2$; $y = -1$; $z = 2$

Distance (–1, –5, –10) (2, –1, 2)

$D = \sqrt{(-3)^2 + (-4)^2 + (-12)^2}$

$D = \sqrt{9 + 16 + 144}$

D = 13 units

14. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$, then find x.

Sol. $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$

$$\sin\left[\sin^{-1}\frac{1}{\sqrt{(x+1)^2+1}}\right] = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$\therefore \quad \sqrt{(x+1)^2+1}=\sqrt{x^2+1}$

$x=-\frac{1}{2}$ which satisfies the above equation.

OR

14. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then find x.

Sol. $\left(\tan^{-1} x\right)^2+\left(\cot^{-1} x\right)^2=\frac{5\pi^2}{8}$

$\left(\tan^{-1} x+\cot^{-1} x\right)^2-2\tan^{-1} x\cot^{-1} x=\frac{5\pi^2}{8}$ $\quad \left[\because a^2+b^2=(a+b)^2-2ab\right]$

$\left(\frac{\pi}{2}\right)^2-2\tan^{-1} x\left(\frac{\pi}{2}-\tan^{-1} x\right)=\frac{5\pi^2}{8}$ $\quad \left[\because \tan^{-1} x+\cot^{-1} x=\frac{\pi}{2}\right]$

Let $\tan^{-1} x = t$

$\frac{\pi^2}{4}-2t\left(\frac{\pi}{2}-t\right)=\frac{5\pi^2}{8}$

$\frac{\pi^2}{4}-\pi t+2t^2=\frac{5\pi^2}{8}$

$2t^2-\pi t+\frac{\pi^2}{4}-\frac{5\pi^2}{8}=0$

$2t^2-\pi t+\frac{2\pi^2-5\pi^2}{8}=0$

$2t^2-\pi t-\frac{3\pi^2}{8}=0$

$\frac{16t^2-8\pi t-3\pi^2}{8}=0$

$16t^2-8\pi t-3\pi^2=0$

$16t^2-12\pi t+4\pi t-3\pi^2=0$

$(4t-3\pi)(4t+\pi)=0$

$t=\frac{3\pi}{4}, t=-\frac{\pi}{4}$

$t=\frac{3\pi}{4}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $\quad t=-\frac{\pi}{4}$

where $t = \tan^{-1} x$

$\tan^{-1} x=-\frac{\pi}{4}$

$x=\tan\left(-\frac{\pi}{4}\right)$

$x=-1$

15. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$, then find $\frac{dy}{dx}$.

Sol. Put $x^2=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1} x^2$

$$y = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

$$y = \tan^{-1}\left[\frac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$y = \frac{\pi}{4}+\theta$$

$$y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2$$

$$\frac{dy}{dx} = 0-\frac{1}{2}\frac{2x}{\sqrt{1-x^4}}$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^4}}$$

16. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol. $x = a\cos\theta + b\sin\theta$ ... (1)

squaring both sides

$x^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta$ ... (2)

$y = a\sin\theta - b\cos\theta$

$y^2 = a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$ ... (3)

(2) + (3)

$x^2 + y^2 = a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta) + 2ab\sin\theta\cos\theta - 2ab\sin\theta\cos\theta$

$\Rightarrow\ x^2 + y^2 = a^2 + b^2$ ... (4)

diff. w.r. to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{... (5)}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y - x\frac{dy}{dx}}{y^2}\right]$$

$$\Rightarrow\ y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx}$$

Hence, $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

17. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

Sol. Let the side of ΔABC be x cm.

Given $\frac{dx}{dt}$ = 2 cm/sec ...(1)

Area of equilateral ΔABC

$$A = \frac{\sqrt{3}}{4}x^2 \qquad ...(2)$$

$\therefore$ Differentiating w.r.t to t

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}2x\frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{2}x\frac{dx}{dt}$$

At $x = 20$ cm, $\frac{dA}{dt} = \frac{\sqrt{3}}{2}(20)\times 2$

$= 20\sqrt{3}\ \text{cm}^2/\text{s}$

18. Find $\int (x+3)\sqrt{3-4x-x^2}\,dx$

Sol. Given $I = \int (x+3)\sqrt{3-4x-x^2}\,dx \qquad ...\text{(i)}$

Let $x+3 = A\frac{d}{dx}(3-4x-x^2)+B$

$x+3 = A(0-4-2x)+B \qquad ...\text{(ii)}$

$= -4A - 2Ax + B$

$x + 3 = -4A + B - 2Ax$

equating coefficient of x and constant term.

$-2A = 1$

$$\boxed{A = -\frac{1}{2}}$$

and $-4A + B = 3$

$+2 + B = 3$

$$\boxed{B = 1}$$

from (ii)

$\therefore$ $(x+3) = -\frac{1}{2}(-4-2x)+1$

from (i)

$$I = \int\left[\left\{-\frac{1}{2}(-4-2x)\sqrt{3-4x-x^2} + \sqrt{3-4x-x^2}\right\}\right]dx$$

$$I = \frac{1}{2}\int(4+2x)\sqrt{3-4x-x^2}\,dx + \int\sqrt{3-4x-x^2}\,dx$$

$I = I_1 + I_2 \qquad ...\text{(iii)}$

Now, $I_1 = \frac{1}{2}\int(4+2x)\sqrt{3-4x-x^2}\,dx$

Put $3 - 4x - x^2 = t$

$-(4 + 2x)\,dx = dt$

$(4 + 2x)\,dx = -dt$

$$I_1 = -\frac{1}{2}\int\sqrt{t}\,dt$$

$$= -\frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} + C_1$$

$$I_1 = -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + C_1 \quad \ldots \text{(iv)}$$

$$I_2 = \int\sqrt{3-4x-x^2}\,dx$$

$$= \int\sqrt{[-(x^2+4x+4-7)}\,dx$$

$$= \int\sqrt{[-\{(x+2)^2-7\}]}\,dx$$

$$= \int\sqrt{7-(x+2)^2}\,dx$$

$$I_2 = \left(\frac{x+2}{2}\right)\sqrt{7-(x+2)^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C_2 \quad \ldots \text{(v)}$$

from (iii), (iv) and (v)

$$I = -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + \left(\frac{x+2}{2}\right)\sqrt{7-(x+2)^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

19. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 etc. The number of articles sold are given below:

School / ***Articles***	***A***	***B***	***C***
Hand fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also find the total funds collected for the purpose.

Write one value generated by the above situation.

Sol. Quantity Martix A = $\begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$

Sell Matrix B = $\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$

Total fund = Quantity matrix × Cost matrix = A × B

$$= \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}_{3\times3} \begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix} 40\times25 & 25\times100 & 35\times50 \\ 50\times25 & 40\times100 & 50\times50 \\ 20\times25 & 30\times100 & 40\times50 \end{bmatrix}$$

$$= \begin{bmatrix} 1000+2500+1750 \\ 1250+4000+2500 \\ 500+3000+2000 \end{bmatrix}$$

$$= \begin{bmatrix} 5250 \\ 7750 \\ 5500 \end{bmatrix}$$

Fund collected by school A = ₹ 5250

Fund collected by school B = ₹ 7750

Fund collected by school C = ₹ 5500

Total fund = 5250 + 7750 + 5550 = ₹ 18550

It shows sympathy and humanity towards flood victims.

20. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R\ (c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

Sol. $(a, b)\ R\ (c, d) \Rightarrow ad(b + c) = bc(a + d)$

(i) Reflexive:

$(a, b)\ R\ (a, b)$ if $ab(b + a) = ba(a + b)$

$ab(a + b) = ab(a + b)$ which is true.

$\therefore$ R is reflexive.

(ii) Symmetric:

Let $(a, b)\ R\ (c, d)$

$\Rightarrow ab(b + c) = bc(a + d)$

or $bc(a + d) = ad(b + c)$

$cb(d + a) = da(c + b)$

$\Rightarrow (c, d)\ R\ (a, b)$

$\therefore$ R is symmetric.

(iii) Transitive:

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow ad(b + c) = bc(a + d)$

and $cf(d + e) = de(c + f)$

$$\frac{b+c}{bc} = \frac{a+d}{ad}$$

and $$\frac{d+e}{de} = \frac{c+f}{cf}$$

$$\frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$$

and $$\frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$$

$$\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$$

and $$\frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f}$$

$$\Rightarrow \frac{1}{a} + \frac{1}{f} = \frac{1}{b} + \frac{1}{e}$$

$be(a + f) = af(b + e)$

$\therefore (a, b)\ R\ (e, f)$. Hence, transitive.

21. Using integration find the area of the triangle formed by postive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Sol. $x^2 + y^2 = 4$ ... (i)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Slope of tangent at $(1, \sqrt{3}) = \frac{-1}{\sqrt{3}}$

equation of tangent is

$$y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1)$$

$$\sqrt{3}y - 3 = -x + 1$$

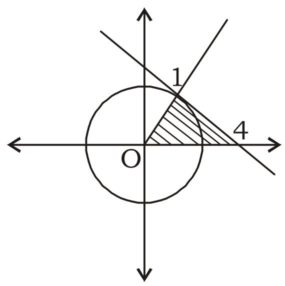
$$x + \sqrt{3}y = 4$$

$$\Rightarrow \quad y = \frac{4 - x}{\sqrt{3}}$$

equation of normal is

$$y - \sqrt{3} = \sqrt{3}(x - 1)$$

$$y = \sqrt{3}x$$



Required Area $= \int_0^1 \sqrt{3}x\,dx + \int_1^4 \frac{4 - x}{\sqrt{3}}dx$

$$= \left[\sqrt{3}\frac{x^2}{2}\right]_0^1 + \frac{1}{\sqrt{3}}\left[4x - \frac{x^2}{2}\right]_1^4$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\left[8 - \frac{7}{2}\right] = 2\sqrt{3} \text{ sq. unit}$$

OR

21. Evaluate $\int_1^3 (e^{2-3x} + x^2 + 1)dx$ as a limit of a sum.

Sol. $I = \int_1^3 (e^{2-3x} + x^2 + 1)dx$

$f(x) = e^{2-3x} + x^2 + 1$ $a = 1;\ b = 3;\ nh = b - a = 2$

$f(1) = e^{-1} + 2$

$f(1 + h) = e^{2-3(1+h)} + (1 + h)^2 + 1$

$= e^{-1}\, e^{-3h} + h^2 + 2h + 2$

$f(1 + 2h) = e^{2-3(1+2h)} + (1 + 2h)^2 + 1$

$= e^{-1}\, e^{-6h} + 4h^2 + 4h + 2$

$f(1 + (n - 1)\ h) = e^{2-3(1+(n-1)h)} + (1 + (n - 1)h)^2 + 1$

$= e^{-1} . e^{-3(n-1)h} + (n - 1)^2\ h^2 + 2(n - 1)h + 2$

On adding,

$f(1) + f(1 + h) + f(1 + 2h) + \ldots\ldots + f[1 + (n - 1)h]$

$= e^{-1}[1 + e^{-3h} + e^{-6h} + \ldots\ldots + e^{-3(n-1)h}] + h^2[1^2 + 2^2 + \ldots\ldots + (n - 1)^2] + 2h[1 + 2 + \ldots\ldots + (n - 1)] + 2n$

$$\therefore \int_1^3 (e^{2-3x} + x^2 + 1)dx = \lim_{h\to 0} h[f(1) + f(1+h) + \ldots\ldots + f(1+(n-1)h)]$$

$$= \lim_{h\to 0} h\left[e^{-1}\left\{\frac{1(1-e^{-3nh})}{1-e^{-3h}}\right\} + \frac{h^2 n(n-1)(2n-1)}{6} + \frac{2hn(n-1)}{2} + 2n \right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{-1}(1-e^{-6})e^{3h}}{e^{3h}-1}\right] + \frac{hn(nh-h)(2nh-h)}{6} + \frac{2nh(nh-h)}{2} + 2nh$$

$$= \lim_{h\to 0} \frac{h[e^{-1}(1-e^{-6})]}{3h\left(\frac{e^{3h}-1}{3h}\right)} + \frac{2(2-0)(4-0)}{6} + \frac{4(2-0)}{2} + 4$$

$$= \frac{e^{-1}-e^{-7}}{3} + \frac{8}{3} + 4 + 4$$

$$= \frac{32}{3} + \frac{e^{-1}-e^{-7}}{3}$$

22. Solve the differential equation: $(\tan^{-1} y - x)\, dy = (1 + y^2)dx$

Sol. $(\tan^{-1} y - x)\, dy = (1 + y^2)dx$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1} y - x}$$

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{1}{1+y^2}.x$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}.x = \frac{\tan^{-1} y}{1+y^2}$$

$$P = \frac{1}{1+y^2} \qquad Q = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

Solution is $(\text{I.F.})x = \int Q(\text{I.F.})dy$

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy \qquad \text{Put } \tan^{-1} y = t \Rightarrow \frac{1}{1+y^2}dy = dt$$

$$xe^{\tan^{-1} y} = \int te^t dt$$

$$xe^{\tan^{-1} y} = te^t - \int e^t dt$$

$$xe^{\tan^{-1} y} = \tan^{-1} y e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

$$\boxed{x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}}$$

OR

22. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ given that $y = 1$, when $x = 0$.

Sol. $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$ ... (i)

$y = 1, \; x = 0$

Put $y = vx$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equation (i) becomes

$v + x\dfrac{dv}{dx} = \dfrac{x(vx)}{x^2 + (vx)^2}$

$x\dfrac{dv}{dx} = \dfrac{v}{1+v^2} - v$

$x\dfrac{dv}{dx} = \dfrac{v - v - v^3}{1+v^2}$

$\dfrac{1+v^2}{v^3}dv = -\dfrac{dx}{x}$

$\int \dfrac{1}{v^3} + \dfrac{1}{v}dv = -\int \dfrac{1}{x}dx$

$\dfrac{v^{-2}}{-2} + \log|v| = -\log|x| + C$

$\dfrac{-x^2}{2y^2} + \log\left|\dfrac{y}{x}\right| = -\log|x| + C$

$\log\left|\dfrac{y}{x}\right| + \log|x| = \dfrac{x^2}{2y^2} + C$

$\log\left|\dfrac{y}{x}\cdot x\right| = \dfrac{x^2}{2y^2} + C$

$\log|y| = \dfrac{x^2}{2y^2} + C$

when $x = 0$, $y = 1$

$\log 1 = 0 + C \Rightarrow C = 0$

solution is $\boxed{\log|y| = \dfrac{x^2}{2y^2}}$

23. If lines $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$ and $\dfrac{x-3}{1} = \dfrac{y-k}{2} = \dfrac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines.

Sol. $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4} = \lambda$ ... (i)

$\dfrac{x-3}{1} = \dfrac{y-k}{2} = \dfrac{z}{1} = \mu$ ... (ii)

from (i) coordinates of general point is

$(2\lambda + 1, 3\lambda - 1, 4\lambda + 1)$

from (ii) coordinates of general point is

$(\mu + 3, 2\mu + k, \mu)$

Lines will intersect

$\Rightarrow \quad 2\lambda + 1 = \mu + 3 \quad \ldots \text{(iii)}$

$3\lambda - 1 = 2\mu + k \quad \ldots \text{(iv)}$

$4\lambda + 1 = \mu \quad \ldots \text{(v)}$

from (iii) and (v)

$2\lambda + 1 = 4\lambda + 1 + 3$

$2\lambda = -3$

$$\lambda = -\frac{3}{2}$$

and $\mu = 4\left(-\frac{3}{2}\right) + 1$

$\mu = -5$

$\therefore$ from (iv)

$$3\left(-\frac{3}{2}\right) - 1 = 2(-5) + k$$

$$\frac{-9-2}{2} = -10 + k$$

$$k = \frac{-11}{2} + 10 = \frac{9}{2}$$

d.r's of given lines are

$< 2, 3, 4 >$ and $< 1, 2, 1 >$

normal vector to plane

$$\vec{n} = \hat{i}(3-8) - \hat{j}(2-8) + \hat{k}(4-3)$$

$$\vec{n} = -5\hat{i} + 2\hat{j} + \hat{k}$$

equation of plane containing these lines

$$\vec{r}\,.\,\vec{n} = \vec{a}\,.\,\vec{n}$$

where $\vec{n} = -5\hat{i} + 2\hat{j} + \hat{k}$

$\vec{a} = \hat{i} - \hat{j} + \hat{k}$

$$\vec{r}\,.\,(-5\hat{i} + 2\hat{j} + \hat{k}) = (-5\hat{i} + 2\hat{j} + \hat{k}).(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r}\,.\,(-5\hat{i} + 2\hat{j} + \hat{k}) = -6$$

$$-5x + 2y + z = -6$$

$$\boxed{5x - 2y - z = 6}$$

24. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then find $P(A)$ and $P(B)$

Sol. $P(\vec{A} \cap B) = \frac{2}{15}$

$$P(B) - P(A \cap B) = \frac{2}{15}$$

$$P(B) - P(A)\,P(B) = \frac{2}{15}$$

$$P(B)\,(1 - P(A)) = \frac{2}{15}$$

$P(A \cap \vec{B}) = \frac{1}{6}$

$$P(A) - P(A \cap B) = \frac{1}{6}$$

$$P(A) - P(A)\,P(B) = \frac{1}{6}$$

$$P(A)\,(1 - P(B)) = \frac{1}{6}$$

$$P(B) = \frac{2}{15(1 - P(A))} \quad \ldots \text{(i)}$$

$$P(A)\left[1 - \frac{2}{15(1 - P(A))}\right] = \frac{1}{6}$$

$$P(A)\left[\frac{15 - 15P(A) - 2}{5(1 - P(A))}\right] = \frac{1}{2}$$

$$P(A)\left[\frac{13 - 15P(A)}{5 - 5P(A)}\right] = \frac{1}{2}$$

$$\frac{13P(A) - 15(P(A))^2}{5 - 5P(A)} = \frac{1}{2}$$

Let $P(A) = t$

$$\frac{13t - 15t^2}{5 - 5t} = \frac{1}{2}$$

$26t - 30t^2 = 5 - 5t$

$30t^2 - 31t + 5 = 0$

$30t^2 - 25t + 6t + 5 = 0$

$5t(6t - 5) - 1(6t - 5) = 0$

$(5t - 1) - (6t - 5) = 0$

$5t - 1 = 0, 6t - 5 = 0$

$$t = \frac{1}{5}, t = \frac{5}{6}$$

where $t = P(A)$ $\qquad$ $t = P(A)$

$P(A) = \frac{5}{6}$ $\qquad$ $P(A) = \frac{1}{5}$

From (i)

$$P(B)(1 - P(A)) = \frac{2}{15}$$

$$P(A) = \frac{1}{5}$$

$$P(B)\left(1 - \frac{1}{5}\right) = \frac{2}{15}$$

$$P(B)\left(\frac{4}{5}\right) = \frac{2}{15\,3}$$

$$P(B) = \frac{1}{6}$$

If $\quad P(A) = \frac{5}{6}$

$$\because \quad P(B)(1 - P(A)) = \frac{2}{15} \Rightarrow P(B)\left(1 - \frac{5}{6}\right) = \frac{2}{15}$$

$$\frac{1}{6}P(B) = \frac{2}{15} \Rightarrow P(B) = \frac{4}{5}$$

25. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values.

Sol. $f(x) = \sin x - \cos x$ $\quad (0 < x < 2\pi)$

$f'(x) = \cos x + \sin x$

$f'(x) = 0 \Rightarrow \cos x = -\sin x$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f''(x) = -\sin x + \cos x$

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$$

$\therefore$ $x = \frac{3\pi}{4}$ is a point of maxima and the maximum value is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$

$$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$$

$\therefore$ $x = \frac{7\pi}{4}$ is a point of minima and the minimum value is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

26. Find graphically, the maximum value of $Z = 2x + 5y$, subject to constraints given below:

$2x + 4y \le 8$

$3x + y \le 6$

$x + y \le 4$

$x \ge 0,\ y \ge 0$

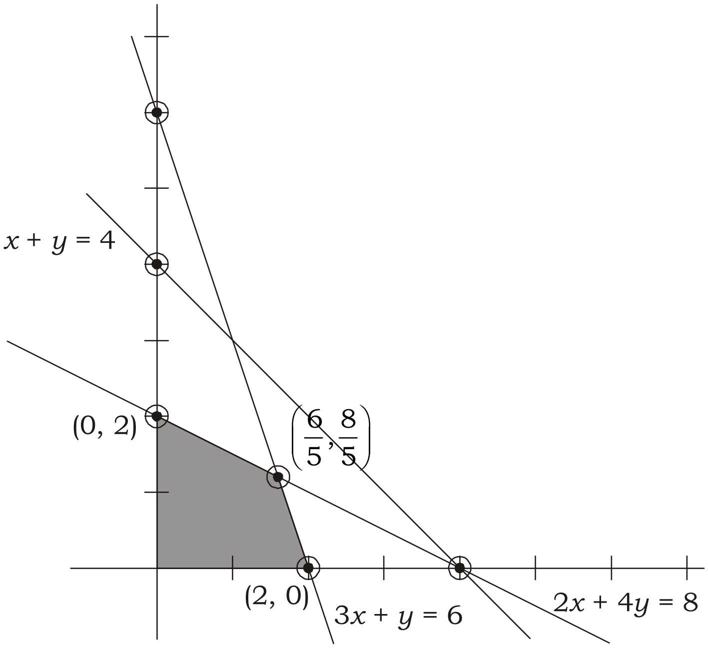
Sol. Maximise $Z = 2x + 5y$

$\frac{x}{4} + \frac{y}{2} = 1$ $\quad$ ...(1)

$3x + y = 6$

$\frac{x}{2} + \frac{y}{6} = 1$ $\quad$ ...(2)

$x \ge 0,\ y \ge 0$ it represents first quadrant



Z at (2, 0) = 2 × 2 + 5 × 0 = 4

Z at (0, 2) = 2 × 0 + 5 × 2 = 10

Z at $\left(\frac{8}{5}, \frac{6}{5}\right) = 2 \times \frac{8}{5} + 5 \times \frac{6}{5} = \frac{16}{5} + 6 = \frac{46}{5}$

∴ Z is maximum at (0, 2)

At $x = 0$, $y = 2$, Z is maximum.

ENJOY MATHS...

Questions in other sets are same but jumbled. If you have any doubt you may call me at

9990088009